NUMERICAL SOLUTIONS OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS USING B-SPLINE COLLOCATION METHODS

A revised synopsis

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1 Introduction

Numerical computations have given a new momentum to the field of problem solving techniques and has achieved a key role in the area of research and development. With rapidly advancing computer systems, improved computational techniques can be proved as of great significance to enhance the correctness and reliability of the computations. Numerical simulations, if carried out properly, can be proved as effective support to physical experiments and theoretical analysis. Moreover, numerical computations can also effectively lead us to the understanding of some crucial areas of pure and applied sciences.

Partial differential equations are used to well describe many natural, physical, chemical and biological phenomena and serve as a tool for the formulation of various scientific and engineering problems. Piecewise polynomial functions have become indispensable tools to approximate the solutions of partial differential equations. One of the most important requirement while employing piecewise polynomials is the smoothness of approximating curve at the joint of the pieces. Spline functions are piecewise polynomial that have high degree of smoothness.

B-splines : The concept of B-spline was first introduced by Schoenberg [1] in 1946, who employed analytic functions while dealing with the problem of approximation of equidistant data. The Basis spline written as B-spline can be described as a smooth piecewise polynomial function. It is a spline function with minimal support concerning a particular degree, smoothness and domain partition.

Let X be a set of N+1 non-decreasing real numbers $x_0 < x_1 < \ldots\ldots < x_{N-1} < x_N$. The x_i's are called knots. The set X is the knot vector. The half open interval $[x_i, x_{i+1})$ is called the i^{th} knot span. If the knots are equally spaced i.e. $x_{i+1} - x_i$ = h = constant for 0 ≤ i ≤ N-1, then the knot vector is said to be uniform; otherwise non-uniform.

The original definition of the B-spline basis function was given by Schumaker [2] using the idea of divided differences. In 1970's, a recurrence relation was independently established by Cox and Boor for the purpose of computing B-splines.

The i^{th} B-spline basis function of degree d, written as $B_{i,d}(x)$, is defined recursively as follows:

$$B_{i,0}(x) = \begin{cases} 1, & \text{if } x_i \leq x < x_{i+1} \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

$$B_{i,d}(x) = \frac{(x - x_i)}{(x_{i+d} - x_i)} B_{i,d-1}(x) + \frac{(x_{i+d+1} - x)}{(x_{i+d+1} - x_{i+1})} B_{i+1,d-1}(x) \tag{2}$$

This formula is known as Cox De Boor's recursion formula.

Properties of B-splines :

1. For $d \geq 0$, $B_{i,d}$ is a non-zero polynomial on $[x_i, x_{i+d+1})$.

2. For $d > 0$, $B_{i,d}$ is a linear combination of two basis functions of degree $(d-1)$.

3. $B_{i,d} = 0$ if $x \notin (x_i, x_{i+d+1})$, i = 0,1,..,$(N-1)$. This is known as the local support property. This property implies that each segment of B-spline curve depends only upon d control points or each control point influences only k curve segments. Due to this property, B-splines are very easy to handle and implement.

4. In a given knot span $[x_i, x_{i+1})$, at most $(d+1)$ of the B-spline basis functions of degree d are non-zero.

5. Non-negativity : For all i, d and x, $B_{i,d}$ is non-negative in the interval $[x_i, x_{i+d+1})$.

6. The relative spacing between the knots is the only factor that influence the shape of the basis function $B_{i,d}$. Translating or scaling the knot vector has no effect on the shape of basis functions and B-spline curves.

7. Partition of unity : Addition of all non-zero d^{th} degree B-splines results in unity i.e. $\sum_{-\infty}^{\infty} B_{i,d} = 1$. This can be proved by using recurrence relation and definition of $B_{i,0}$.

8. For a given knot sequence, if B-splines of degree d do not vanish over a particular knot sequence, then they are linearly independent.

9. The minimal support for $B_{i,d}$ for a given degree, smoothness and domain partition is $[x_i, x_{i+d+1}]$.

10. Derivative of B-spline is given as

$$\frac{d}{dx}B_{i,d}(x) = \frac{d}{x_{i+d} - x_i}B_{i,d-1}(x) - \frac{d}{x_{i+d+1} - x_{i+1}}B_{i+1,d-1}(x) \tag{3}$$

For a non-decreasing knot sequence $\{x_i\}_{i=1}^{\infty}$, the zeroth degree B-spline is defined as

$$B_{i,0}(x) = \begin{cases} 1, & \text{if } x_i \leq x < x_{i+1} \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

The first B-spline of degree $B_{i,1}$ is called linear B-spline:

$$B_{i,1}(x) = \begin{cases} \frac{(x-x_i)}{(x_{i+1}-x_i)}, & x \in [x_i, x_{i+1}) \\ \frac{(x_{i+2}-x)}{(x_{i+2}-x_{i+1})}, & x \in [x_{i+1}, x_{i+2}) \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

Cubic B-splines : Explicitly, third degree B-splines termed as cubic B-splines is given as

$$B_{i,3}(x) = \frac{1}{h^3}\begin{cases} (x - x_{i-2})^3 & x \in [x_{i-2}, x_{i-1}) \\ (x - x_{i-2})^3 - 4(x - x_{i-1})^3 & x \in [x_{i-1}, x_i) \\ (x_{i+2} - x)^3 - 4(x_{i+1} - x)^3 & x \in [x_i, x_{i+1}) \\ (x_{i+2} - x)^3 & x \in [x_{i+1}, x_{i+2}) \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

The values of $B_{i,3}(x)$ and its derivatives at the nodal points are given in Table 1.

Table 1: Values of $B_{i,3}(x)$ and its derivatives at the nodal points

	x_{i-2}	x_{i-1}	x_i	x_{i+1}	x_{i+2}
$B_i(x)$	0	1	4	1	0
$B_i^{'}(x)$	0	$3/h$	0	$-3/h$	0
$B_i^{''}(x)$	0	$6/h^2$	$-12/h^2$	$6/h^2$	0

A variety of problems arising in the field of engineering, mathematics and applied sciences are dealt with successful implementation of splines. Function approximations, integro-differential equations,

data fitting, Computer-Aided Geometric Designs (CAGD), wavelets etc. are some of them. A distinguished work has been done in some of the research papers that contributed to the development and advancement of splines. Books by De Boor [3], Prenter [4], etc. have made a significant contribution in the evolvement and use of splines.

The purpose of this study is to develop new and novel numerical methods or variants of collocation method with cubic B-splines as basis functions to tackle linear and nonlinear initial and/or boundary value problems. Our main aim resides in exploring the scope of finding accurate and reliable solutions to both linear and nonlinear partial differential equations in one and two dimensions.

2 Methodology

2.1 Collocation Method

Collocation method is a higher order numerical discretization technique that provides very accurate solutions to differential equations with relatively less computational effort. With less sophisticated techniques, this method is more explicit, simple to use and implement in many practical problems especially for nonlinear cases.

Collocation methods will be developed with B-spline basis functions of degree three over uniform meshes. The method will be applied and tested on different single and coupled partial differential equations in one and two dimensions. The approximations to the solutions of the partial differential equations are expected to get the desired results. Numerical schemes are expected to involve relatively small number of arithmetic operations and minimal computations. The problems to be dealt in this work arise in the mathematical modeling of many physical, chemical and biological phenomena and are related to the field of engineering and applied sciences. Collocation methods have an edge over various methods as they approximate the solutions at all points in the domain rather than the grid points. Smoothness and compact support properties of B-splines as basis functions reduce the arithmetic to a great extent thus making the method more efficient with minimal storage, cost and computational efforts.

Collocation method approximates the solution of differential equations in some form of a linear combination of basis functions. Consider a differential equation $f(x, u, u_x, u_{xx})$=0 to be solved by collocation method. Let the function is defined in the domain [a, b] with boundary conditions given by u(a)=f_0, u(b)=f_1.

This method begins with a proper choice of basis functions $\phi_1, \phi_2, \ldots\ldots, \phi_n$ and a set of points $a = x_0 < x_1 < \ldots\ldots < x_{N-1} < x_N = b$ called nodes or collocation points in the domain [a, b]. The approximate solution can be written in the form of

$$U = \sum_{i=0}^{N} c_i \phi_i(x) \tag{7}$$

Here N can be thought as of quality parameter. As N increases, the error in the approximation must reduce. This method requires that this approximation satisfies the given differential equation at each of the nodes and also satisfies the boundary conditions.

For this the residual must be zero at the selected points. The residual is defined as $r(x, u, u_x, u_{xx})$ = $f(x, u, u_x, u_{xx})$ – $f(x, U, U_x, U_{xx})$, where U is the approximate solution.

Much attention has been given in the literature for the choice of the knots x_i and the basis functions ϕ_i and it has been found that spline curves or piecewise polynomials are more effective in representing the solution of the differential equations than pure polynomials.

2.2 B-spline Collocation Method

B-spline collocation method is a method for finding the numerical solutions of differential equations. The idea is to choose a finite dimensional space of candidate solutions (B-splines upto a certain degree) and a number of points within the domain (called collocation points), and to select that solution which satisfies the given equation at each collocation point.

We consider the basis functions for which the knots are equidistant. In such a case, the B-spline is said to be uniform. A fundamental theorem states that every spline function of a given degree, smoothness, and domain partition can be represented as a linear combination of B-splines of the same degree and smoothness, and over the same partition. This attribute immediately implies the banded structure of matrices occuring in interpolation and collocation problems. As a result, these matrices can be efficiently stored in a computer and inverted by Gaussian elimination without pivoting. The problem of stable calculation of polynomial B-splines and their derivatives may be found in Prenter [4].

2.3 Cubic B-spline Collocation Method

Let us consider a uniform partition of the domain Ω by the knots x_j, j = 0,1,2,....,N, such that h = x_j - x_{j-1} is the length of each interval.

In cubic B-spline collocation method, exact solution u(x,t) is approximated by S(x,t) in the form:

$$S(x,t) = \sum_{j=-1}^{N+1} a_j(t) B_j(x), \tag{8}$$

where $a_j(t)$'s are unknown time dependent quantities which we find using boundary conditions and collocation method. $B_{i,3}(x)$ is cubic B-spline given by the eq. (6) in which $[B_{-1}(x), B_0(x), B_1(x), ..., B_N(x), B_{N+1}(x)]$ forms a basis over the interval. It is assumed that S(x,t) satisfies following interpolatory and end conditions -

$$S(x_j,t) = u(x_j,t), \qquad 0 \le j \le N \tag{9}$$

$$S"(x_j,t) = u"(x_j,t) - \frac{1}{12}h^2 u^{(4)}(x_j,t), \qquad j = 0, N \tag{10}$$

If u(x,t) is sufficiently smooth and S(x,t) is a unique cubic spline interpolant satisfying above boundary conditions then we have, from [5], :

$$S'(x_j,t) = u'(x_j,t) + O(h^4), \qquad 0 \le j \le N \tag{11}$$

$$S"(x_j,t) = u"(x_j,t) - \frac{1}{12}h^2 u^{(4)}(x_j,t) + O(h^4), \quad 0 \le j \le N \tag{12}$$

The approximate values S(x,t) and its first order derivatives at the knots are defined using Taylor expansions and finite difference as follows :

For j = 0,

$$u^{(4)}(x_j,t) = \frac{2S"(x_0,t) - 5S"(x_1,t) + 4S"(x_2,t) - S"(x_3,t)}{h^2} + O(h^2) \tag{13}$$

For 1 ≤ j ≤ N-1,

$$u^{(4)}(x_j,t) = \frac{S"(x_{j-1},t) - 2S"(x_j,t) + S"(x_{j+1},t)}{h^2} + O(h^2) \tag{14}$$

For j = N,

$$u^{(4)}(x_j,t) = \frac{2S"(x_N,t) - 5S"(x_{N-1},t) + 4S"(x_{N-2},t) - S"(x_{N-3},t)}{h^2} + O(h^2) \tag{15}$$

Using equations (13), (14) and (15) in (11), (12), we get

$$u'(x_j,t) = S'(x_j,t) + O(h^4), 0 \le j \le N \tag{16}$$

For j = 0,

$$u"(x_0,t) = \frac{14S"(x_0,t) - 5S"(x_1,t) + 4S"(x_2,t) - S"(x_3,t)}{12} + O(h^4) \tag{17}$$

For 1 ≤ j ≤ N-1,

$$u"(x_j,t) = \frac{S"(x_{j-1},t) + 10S"(x_j,t) + S"(x_{j+1},t)}{12} + O(h^4) \tag{18}$$

For j = N,

$$u"(x_N,t) = \frac{14S"(x_N,t) - 5S"(x_{N-1},t) + 4S"(x_{N-2},t) - S"(x_{N-3},t)}{12} + O(h^4) \tag{19}$$

Using equations (6) and (8), equations (17), (18), (19) can be simplified to be written as

For j = 0,

$$u"(x_0,t) = \frac{14a_{-1} - 33a_0 + 28a_1 - 14a_2 + 6a_3 - a_4}{2h^2} \tag{20}$$

For 1 ≤ j ≤ N-1,

$$u"(x_j,t) = \frac{a_{j-2} + 8a_{j-1} - 18a_j + 8a_{j+1} + a_{j+2}}{2h^2} \tag{21}$$

For j = N,

$$u"(x_N,t) = \frac{14a_{N+1} - 33a_N + 28a_{N-1} - 14a_{N-2} + 6a_{N-3} - a_{N-4}}{2h^2} \tag{22}$$

3 Error Norms and Order of Convergence

The performance of the proposed methods needs to be checked in terms of accuracy and efficiency. To confirm the efficiency of present methods, numerical approximation errors have been computed using the following formulae:

While dealing with one-dimensional problems, let $u(x_i,t)$ be the exact solution and $U(x_i,t)$ be the approximate solution at the grid point x_i obtained by the proposed method at time t, then

$$L_\infty(u) = \max_{1\le i\le N} |u(x_i,t) - U(x_i t)| \tag{23}$$

and

$$L_2(u) = \sqrt{\sum_{i=1}^{i=N} |u(x_i,t) - U(x_i t)|^2} \tag{24}$$

$$GRE = \frac{\sum_i |u(x_i,t) - U(x_i,t)|}{\sum_i |U(x_i,t)|} \tag{25}$$

While dealing with two-dimensional problems, let $u(x_i,y_j,t)$ be the exact solution and $U(x_i,y_j,t)$ be the approximate solution at the grid point (x_i,y_j) obtained by the proposed method at time t, then

$$L_\infty(u) = \max_{1\le i\le N, 1\le j\le M} |u(x_i,y_j,t) - U(x_i,y_j,t)| \tag{26}$$

and

$$L_2(u) = \sqrt{\sum_{i=1,j=1}^{i=N,j=M} |u(x_i, y_j, t) - U(x_i, y_j, t)|^2} \tag{27}$$

The numerical order of convergence R for the proposed method can be obtained using the following formulation

$$R = \frac{log\left(\frac{Error(N_1)}{Error(N_2)}\right)}{log\left(\frac{N_1}{N_2}\right)} \tag{28}$$

where $Error(N_1)$ and $Error(N_2)$ are the L_∞ errors at number of partitions N_1 and N_2 respectively.

4 Literature Review

Collocation method has become increasingly popular technique to solve a variety of linear and nonlinear partial differential equations. Cubic B-spline collocation method was used by Goh et al. [7] to solve heat and advection diffusion equations in one dimension. Dag and Saka [8] used B-spline collocation method for equal width equation. B-spline collocation method was also used by Kadalbajoo and Arora [9] to deal with the singular perturbation problems and by Zahra [10] to study PHI-four and AllenCahn equations. Ersoy and Dag [11] applied this method to solve Kuramoto Sivashinsky equation. Khater et al. [12] obtained numerical solution of the Burgers'-type equations by using cubic spline collocation method.

The Burgers'–Fisher equation has many applications in the fields such as gas dynamics, number theory, heat conduction, elasticity, etc. Various numerical and analytical methods have been used by various researchers to deal with the Burgers'-Fisher equation. Zhu and Kang [13] investigated the solution of the Burgers'–Fisher equation by using the B-spline quasi-interpolation method. Sari et al. [14] proposed a compact finite difference method for the generalized Burgers'–Fisher equation. Recently Yadav and Jiwari [15] employed Galerkin's finite element method to analyse and approximate the Burgers'-Fisher equation. S Malik, Qureshi, Amir, A Malik and Haq [16] used exp-function method hybridized with heuristic computation for the numerical simulation of the Burgers'-Fisher equation.

The Burgers'–Huxley equation was investigated by Satsuma et al. [6] in 1987. In 2004, Ismail et al. [17] proposed the Adomian decomposition method for Burgers'–Huxley and Burgers'–Fisher equations. In 2006, Javidi [18] presented a spectral collocation method for finding numerical solutions of the generalized Burgers'–Huxley equation. In 2009, Mittal and Jiwari [19] obtained an efficient numerical solution of the Burgers'–Huxely equation by using the method of differential quadrature. In 2010, Tomasiello [20] solved the Burgers'–Huxley equation by using a generalized version of the Iterative Differential Quadrature method. In 2012, Çelik [21] proposed an efficient numerical method for the solution of nonlinear partial differential equations using the Haar wavelets approach and tested it on an generalized Burgers'–Huxley equation. In the same year 2012, Dehghan et al. [22] proposed three efficient methods to obtain numerical solution of the generalized Burgers'–Huxley equation.

A reaction-diffusion equation in two dimensions can be used to model many natural phenomena and population evolution in a two-dimensional territory that can possibly be nonlinear in nature, taking in consideration, the harmony of reaction and diffusion. An enormous amount of research work has been done to solve Fisher's reaction-diffusion equation in one and two dimensions. A Galerkin finite element method has been proposed by Tang and Weber [23] and a finite difference scheme has been discussed by Mickens [24] to numerically approximate the Fisher's equation. A comparative study has been conducted by Rizwan [25] to compare non-standard finite difference schemes with nodal integral method. Mittal and Kumar [26] applied wavelet Galerkin method to deal with Fisher's equation.

Telegraph equation is found to be much more appropriate than ordinary diffusion equations for modeling of reaction-diffusion phenomenon. The vibration of structures, for eg. buildings and machines, is modelled through hyperbolic partial differential equations, which are the basis of atomic physics. Problems of random walk theory, propagation and transmission of signal are significantly modelled through telegraph equation. In the recent years, many numerical methods e.g. finite element, finite difference and collocation methods have been developed to solve the linear and nonlinear telegraph equations. One dimensional telegraph equation of order two has been dealt with by Dehghan and Ghesmati [27] using DRBIE method and Chebyshev cardinal functions. Dehghan and

Ghesmati [28] developed two mesh-less methods, MLPG and MLWS to solve the telegraph equation in two dimensions. Mohebbi and Dehghan [29] gave a compact finite difference approximation of fourth order in space and used collocation method for solving two dimensional linear hyperbolic telegraph equation. In [30], Mohanty and Jain introduced a new unconditionally stable alternating direction implicit (ADI) scheme of second order accurate for two-dimensional telegraph equation. The solution is progressed in time direction by splitting the systems in x- and y-directions and solving two systems for each time level.

In this work, we will be dealing with the numerical solutions of partial differential equations such as Burgers'-Fisher equation, Burgers'-Huxley equation, two dimensional Fisher's reaction-diffusion equation, one and two dimensional hyperbolic telegraph equation. We will develop cubic B-spline collocation method or its variants to find the numerical approximations to solutions of these equations. In order to demonstrate the effectiveness of the method, numerical experiments will be performed on various examples. The stability of the method will be discussed. The proposed method is expected to be accurate, reliable and efficient.

5 Numerical problems

5.1 Numerical Solutions of Nonlinear Parabolic Partial Differential Equations

Problem 1 : One Dimensional Burgers'-Fisher Equation

We consider the following Burgers'-Fisher equation of the form

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + \alpha u \frac{\partial u}{\partial x} + \beta u(1-u) = 0, \qquad (x,t) \in \Omega \times (0,T) \tag{29}$$

where α and β are advection and source/sink constants and $\Omega \subset \mathbb{R}$.

Initial and boundary conditions are as follows

$$u(0,x) = u_0(x) \qquad for \;\; x \in \Omega \tag{30}$$

$$u = 0 \qquad\qquad on \;\; (0,T) \times \partial\Omega \tag{31}$$

where the initial data $u_0(x)$ is assumed to be non-smooth.

This problem deals with the numerical solution of Burgers'-Fisher equation, which is nonlinear and parabolic in nature. It describes the mathematical model of many physical situations occurring in various fields of science and engineering such as heat conduction, gas dynamics, chemical physics, nonlinear optics etc. For example, it models velocity profile for viscous fluid in fluid dynamics, gas dynamics in an exhaust pipe etc. It represents a prototypical model for relating the interaction between the convection effect, reaction mechanism and diffusion transport. It plays a significant role in nonlinear physics and thus has a great practical importance.

The aim of this work is to investigate the numerical solutions of the Burgers'-Fisher equation for different parametric values using collocation method with cubic B-splines as basis functions. The stability of the method will be discussed using Fourier series analysis (von Neumann) method. In order to demonstrate the effectiveness of the scheme, numerical experiments will be performed on various examples. The proposed scheme is expected to be viable and accurate.

Problem 2 : One Dimensional Burgers'-Huxley Equation

Behaviors of many physical systems encountered in models of reaction mechanisms, convection effects, and diffusion transports give rise to the Burgers'-Huxley equation. The following Burgers'-Huxley equation problem arising in various fields of science will be considered:

$$\frac{\partial u}{\partial t} + \alpha u^{\delta}\frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = \beta u(1-u^{\delta})(u^{\delta}-\gamma), \quad 0 \le x \le 1, \quad t \ge 0 \tag{32}$$

with the initial condition

$$u(x,0) = f(x) \tag{33}$$

and boundary conditions

$$u(a,t) = g_0(t), \qquad u(b,t) = g_1(t) \tag{34}$$

where α, β, γ and δ are parameters, $\beta \geq 0$, $\delta > 0$, $\gamma \in (0,1)$.

Burger's–Huxley equation shows a prototype model for describing the interaction between reaction mechanisms, convection effects and diffusion transports. The equation was investigated by Satsuma et al. [6] in 1986. Approximate solutions of nonlinear differential equations are of importance in physical problems. So far there exists no general method for finding solutions of nonlinear diffusion equations.

Problem 3 : Two Dimensional Fisher's Reaction-Diffusion Equation

A reaction-diffusion equation in two dimensions can be used to model many natural phenomena and population evolution in a two-dimensional territory that can possibly be nonlinear in nature, taking in consideration, the harmony of reaction and diffusion. Fisher's equation with appropriate initial and boundary conditions is used to model various phenomena like spreading of bushfires and epidemics, auto-catalytic Brownian motion, spread of invasive species, bacteria and epidemics etc.

This problem revolves around the Fisher's reaction-diffusion equation in two dimensions given as

$$\frac{\partial u}{\partial t} = l_1 \frac{\partial^2 u}{\partial x^2} + l_2 \frac{\partial^2 u}{\partial y^2} + \mu f(u). \tag{35}$$

where $f(u)$ is the reaction term that is nonlinear, l_1 and l_2 are diffusive coefficients and μ is reactive factor.

With initial conditions

$$u(x, y, 0) = u_0(x, y), \qquad 0 \leq u_0 \leq 1, \qquad (x, y) \in \Omega_0 \subset \Omega. \tag{36}$$

and boundary conditions

$$\mathbf{n} \cdot \nabla u(x, y, t) = 0, \qquad t \geq 0, \qquad (x, y) \in \partial\Omega \tag{37}$$

or

$$u(x, y, t) = f_0, \qquad t \geq 0, \qquad (x, y) \in \partial\Omega \tag{38}$$

where $\Omega = \{(x, y) : |x| < p, |y| < q\}, \partial\Omega = \{(x, y) : |x| = p, |y| = q\}$, ∇ is the two dimensional gradient operator, $\mathbf{n}$ is unit normal vector on $\partial\Omega$, Ω has a local subset Ω_0 that confirms the local support of the initial disturbance $u_0(x, y)$. If Ω_0 is small enough as compared to Ω, and t is not large enough, then we will use the boundary condition given in Eq. (38). Otherwise the natural boundary condition will be used.

A reaction-diffusion equation in one dimension approximately describe the response in a very large scale to any local initial disturbance, which possess long term traveling-wave solution for many choices of $f(u)$, say $f(u) = u^n(1 - u)$, $n = 1, 2$. Here, we have considered $f(u) = u(1 - u)$. We expect a local disturbance to expand completely and after a long time duration, to turn into a quasi-traveling wave along with a circular wave front, depending on the asymptotic analysis. Its speed and direction changes with distance because of two-dimensional effect.

5.2 Numerical Solutions of Nonlinear Hyperbolic Differential Equations

Problem 4 : One Dimensional Wave, Telegraph and Heat Conduction Equation

In this problem, we are concerned with the numerical solution of the telegraph equation which is widely used in many different fields of science and mathematical engineering such as vibration of structures, random walk theory and transmission and propagation of electric signal. The telegraph equation, also known as damped wave equation, governs physically the voltage and current on an electrical transmission line with distance and time. The equations come from Oliver Heaviside who in the 1880s developed the transmission line model. The model demonstrates that the electromagnetic waves can be reflected on the wire, and that wave patterns can appear along the line. The theory applies to transmission lines of all frequencies including high-frequency transmission lines (such as telegraph wires and radio frequency conductors), audio frequency (such as telephone lines), low frequency (such as power lines) and direct current.

We consider one dimensional linear and nonlinear second order time dependent hyperbolic partial differential equations of the form

$$\frac{\partial^\alpha u}{\partial t^\alpha} + (\alpha - 1)A(x, t)\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + (\alpha - 2)B(x, t)\frac{\partial u}{\partial x} + C(x, t)u + f\big(x, t, u, \frac{\partial u}{\partial x}\big) \tag{39}$$

with initial conditions

$$u(x,0) = g_1(x), \qquad (\alpha-1)\frac{\partial u}{\partial t}(x,0) = (\alpha-1)g_2(x) \tag{40}$$

and boundary conditions

$$u(0,t) = f_1(t), \qquad u(1,t) = f_2(t) \tag{41}$$

where g_1 and g_2 are prescribed functions, $\alpha = 1$ or 2 and A, B, C are continuous functions or finite constants. The case $\alpha = 1$ corresponds to the reaction problem for B = 0, the diffusion–convection problem for C = 0, and heat equation for B = C = 0 respectively. The case $\alpha = 2$ corresponds to the telegraph problem.

We will also consider the one dimensional wave equation for heat conduction

$$\epsilon\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, \quad t > 0, \tag{42}$$

with initial conditions

$$u(x,0) = g(x), \qquad \frac{\partial u}{\partial t}(x,0) = \mu(x) \quad 0 \le x \le 1 \tag{43}$$

and boundary conditions

$$u(0,t) = 0, \qquad u(1,t) = 0, \quad t > 0. \tag{44}$$

This problem will deal with the numerical solutions of one dimensional wave, telegraph and heat conduction equations. Hyperbolic diffusion models have been widely discussed in literature since they overcome the unphysical property of infinite speed of propagation that is specific to parabolic models. There are experimental evidences indicating that diffusive process takes place with finite velocity. So, the wave nature of diffusive processes needs to be taken into account, although it can be ignored in some applications. Hyperbolic partial differential equations are used to formulate various fundamental concepts in atomic physics, aerospace engineering, industry, biology and chemistry.

Problem 5 : Two Dimensional Hyperbolic Telegraph Equation

We consider the second order two dimensional hyperbolic telegraph equation :

$$\frac{\partial^2 u}{\partial t^2} + 2\alpha \frac{\partial u}{\partial t} + \beta^2 u - f(x, y, t) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \tag{45}$$

where $\Omega \in [0 < \text{x,y} < 1] \times [\text{t} > 0]$.

The hyperbolic differential equations are the basis for fundamental equations of atomic physics and commonly used in signal analysis for transmission and propagation of electrical signals. Eq. (45) is damped wave equation for $\alpha > 0$, $\beta = 0$ of which the solution is of great importance in wave phenomena and telegraph equation for $\alpha \geq \beta > 0$. The initial conditions are assumed to be

$$u(x, y, 0) = \phi_1(x, y), \quad 0 < x, y < 1$$

$$\frac{\partial u}{\partial t}(x, y, 0) = \phi_2(x, y), \quad 0 < x, y < 1 \tag{46}$$

and Dirichlet's boundary conditions

$$u(x, y, t) = p_1(x, y, t), \quad at \;\; x, y = \pm 1, t \geq 0,$$

or

$$u_x(x, y, t) = p_2(x, y, t), \quad u_y(x, y, t) = p_3(x, y, t) \quad at \;\; x, y = \pm 1, t \geq 0. \tag{47}$$

The functions f(x,y,t), ϕ_1(x, y), ϕ_2(x, y), p_1(x, y, t), p_2(x, y, t), p_3(x, y, t) are defined and continuous on Ω.

6 Objectives

1. To develop novel numerical techniques for approximating the solutions of linear and nonlinear partial differential equations in one and two dimensions.

2. To provide highly accurate and efficient numerical solutions to parabolic and hyperbolic par-

tial differential equations in one and two dimensions.

3. Providing reliable solutions along with minimal computational efforts, least arithmetic operations and less CPU time.

4. To develop numerical schemes that can deal with nonlinear partial differential equations of higher dimensions.

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